

ON APPROXIMATELY LEFT ϕ -BIPROJECTIVE BANACH ALGEBRAS

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ABSTRACT. In this paper, for a Banach algebra A , we introduced the new notions of approximately left ϕ -biprojective and approximately left character biprojective, where ϕ is a non-zero multiplicative linear functional on A . We show that for SIN group G , Segal algebra $S(G)$ is approximately left ϕ_1 -biprojective if and only if G is amenable, where ϕ_1 is the augmentation character on $S(G)$. Also we showed that the measure algebra $M(G)$ is approximately left character biprojective if and only if G is discrete and amenable. For a Clifford semigroup S , we show that $\ell^1(S)$ is approximately left character biprojective if and only if $\ell^1(S)$ is pseudo-amenable. We study the hereditary property of these new notions. Finally we give some examples among semigroup algebras and Triangular Banach algebras to show the differences of these notions and the classical ones.

1. INTRODUCTION AND PRELIMINARIES

The class of amenable Banach algebras has been introduced by Johnson. A Banach algebra A is called amenable if for every continuous derivation $D : A \rightarrow X^*$ there exists $x_0 \in X^*$ such that

$$D(a) = a \cdot x_0 - x_0 \cdot a \quad (a \in A).$$

He also showed that A is amenable if and only if there exists a bounded net (m_α) in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0, \quad \pi_A(m_\alpha)a \rightarrow a \quad (a \in A),$$

where $\pi_A : A \otimes_p A \rightarrow A$ is given by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$, see [11]. There is another approach to study Banach algebra through the homological theory. Two important notions of biflatness and biprojectivity for Banach algebras have key role in homological theory. In fact a Banach algebra A is called biflat (biprojective), if there exists a bounded A -bimodule morphism $\rho : A \rightarrow (A \otimes_p A)^{**}$ ($\rho : A \rightarrow A \otimes_p A$) such that $\pi_A^{**} \circ \rho$ is the canonical embedding of A into A^{**} (ρ is a right inverse for π_A), respectively see [8]. Note that a Banach algebra A is amenable if and only if A is biflat and A has a bounded approximate identity. In fact the dual notion for amenability in Banach homology is biflatness. It is known that for a locally compact group G , $L^1(G)$ is biflat (biprojective) if and only if G is amenable(compact), respectively.

Recently a notion of amenability related to a character has been introduced in [12]. Indeed a Banach algebra A is called left ϕ -amenable, if there exists a bounded net (a_α) in A such that $aa_\alpha - \phi(a)a_\alpha \rightarrow 0$ and $\phi(a_\alpha) \rightarrow 1$ for all $a \in A$, where $\phi \in \Delta(A)$. For a locally compact group G , the Fourier algebra $A(G)$ is always left ϕ -amenable. Also the group algebra $L^1(G)$ is left ϕ -amenable if and only if G is amenable. For the further information about this notion see [23] and [2]. Motivated by these consideration, author

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with A. Pourabbas introduced notions of homological algebra theory related to a character. A Banach algebra A is called ϕ -biflat(ϕ -biprojective) if there exists a bounded A -bimodule morphism

$$\rho : A \rightarrow (A \otimes_p A)^{**} (\rho : A \rightarrow A \otimes_p A)$$

such that

$$\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a) (\phi \circ \pi_A \circ \rho(a) = \phi(a)) \quad (a \in A),$$

respectively, where $\tilde{\phi}(F) = F(\phi)$ for all $F \in A^{**}$. For a locally compact group G , we showed that Segal algebra $S(G)$ is ϕ -biflat(biprojective) if and only if G is amenable(compact). Also $A(G)$ is ϕ -biprojective if and only if G is discrete, see [19] and [21].

Recently approximate versions of amenability and homological properties of Banach algebras have been under more observations. In [24] Zhang introduced the notion of approximately biprojective Banach algebras, that is, A is approximately biprojective if there exists a net of A -bimodule morphism $\rho_\alpha : A \rightarrow A \otimes_p A$ such that

$$\pi_A \circ \rho_\alpha(a) \rightarrow a \quad (a \in A).$$

Author with A. Pourabbas investigated approximate biprojectivity of some semigroup algebras and some related Triangular Banach algebras see [20] and [22]. Approximate amenable Banach algebras have been introduced by Ghahramani and Loy. Indeed a Banach algebra A is approximate amenable if for every continuous derivation $D : A \rightarrow X^*$, there exists a net (x_α) in X^* such that

$$D(a) = \lim_{\alpha} a \cdot x_\alpha - x_\alpha \cdot a \quad (a \in A).$$

Other extensions of amenability are pseudo-amenability and pseudo-contractibility. A Banach algebra A is pseudo-amenable (pseudo-contractible) if there exists a not necessarily bounded net (m_α) in $A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0, \quad (a \cdot m_\alpha = m_\alpha \cdot a), \quad \pi_A(m_\alpha)a \rightarrow a \quad (a \in A),$$

respectively. For more information about these concepts the reader referred to [7], [5] and [6]. Motivated by these considerations, in [1] the approximate notions of amenability have been introduced and studied. A Banach algebra A is called approximately left ϕ -amenable if there exists a (not necessarily bounded) net (a_α) in A such that $aa_\alpha - \phi(a)a_\alpha \rightarrow 0$ and $\phi(a_\alpha) \rightarrow 1$ for all $a \in A$. Also A is approximately character amenable, if A is approximately left ϕ -amenable for all $\phi \in \Delta(A) \cup \{0\}$. They showed that $L^1(G)^{**}$ is character amenable if and only if G is discrete and amenable. Also they showed that $M(G)$ is character amenable if and only if G is discrete and amenable.

In this paper, we extend the notions of ϕ -biflatness and ϕ -biprojectivity. We give an approximate notion of homological algebra related to approximate left ϕ -amenability. Here the definition of our new notion:

Definition 1.1. Let A be a Banach algebra and $\phi \in \Delta(A)$. A is called approximately left ϕ -biprojective if there exists a net of bounded linear maps from A into $A \otimes_p A$, say $(\rho_\alpha)_{\alpha \in I}$, such that

- (i) $a \cdot \rho_\alpha(x) - \rho_\alpha(ax) \xrightarrow{\|\cdot\|} 0$;
- (ii) $\rho_\alpha(xa) - \phi(a)\rho_\alpha(x) \xrightarrow{\|\cdot\|} 0$;
- (iii) $\phi \circ \pi_A \circ \rho_\alpha(x) - \phi(x) \rightarrow 0$,

for every $a, x \in A$. We say that A is approximately left character biprojective if A is approximately left ϕ -biprojective for all $\phi \in \Delta(A)$.

We show that approximate left ϕ -amenability gives approximate left ϕ -biprojectivity. We study the hereditary properties of this new notion. We show that for SIN group G , Segal algebra $S(G)$ is approximately left ϕ_1 -biprojective if and only if G is amenable, where ϕ_1 is augmentation character on $S(G)$. Also we showed that the measure algebra $M(G)$ is approximately left character biprojective if and only if G is discrete and amenable. We give some Banach algebra among Triangular Banach algebras which is never approximately left ϕ -biprojective. Also we give some examples which reveal the differences our new notions and the classical ones.

We remark some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. Throughout this paper the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A . Let A be a Banach algebra. The projective tensor product $A \otimes_p A$ is a Banach A -bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Let A and B be Banach algebras and $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. We denote $\phi \otimes \psi$ for a map defined by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for all $a \in A$ and $b \in B$. It is easy to see that $\phi \otimes \psi \in \Delta(A \otimes_p B)$.

Let A and B be a Banach algebras and let X be a Banach A, B -module, that is, X is a Banach space, a left A -module and a right B -module with the compatible module action that satisfies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $\|a \cdot x \cdot b\| \leq \|a\| \|x\| \|b\|$ for every $a \in A, x \in X, b \in B$. With the usual matrix operation and $\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|x\| + \|b\|$, $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ becomes a Banach algebra which is called Triangular

Banach algebra. Let $\phi \in \Delta(B)$. We define a character $\psi_\phi \in \Delta(T)$ via $\psi_\phi \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \phi(b)$ for every $a \in A, b \in B$ and $x \in X$.

For a locally compact group G , $M(G)$ is denoted for measure algebra and $A(G)$ is denoted for Fourier algebra.

2. APPROXIMATE LEFT ϕ -BIPROJECTIVITY

In this section we study the general properties of approximately left ϕ -biprojective Banach algebras.

Proposition 2.1. *Let A be a Banach algebra with $\phi \in \Delta(A)$. Suppose that A is an approximately left ϕ -biprojective Banach algebra which has an element a_0 such that $aa_0 = a_0a$ and $\phi(a_0) = 1$. Then A is approximately left ϕ -amenable.*

Proof. Let $(\rho_\alpha)_{\alpha \in I}$ be as in Definition 1.1. Let a_0 be an element in A such that $aa_0 = a_0a$ and $\phi(a_0) = 1$. Set $n_\alpha = \rho_\alpha(a_0)$. It is clear that (n_α) is a net in $A \otimes_p A$ such that

$$\begin{aligned} a \cdot n_\alpha - \phi(a)n_\alpha &= a \cdot \rho_\alpha(a_0) - \phi(a)\rho_\alpha(a_0) \\ &= a \cdot \rho_\alpha(a_0) - \rho_\alpha(aa_0) + \rho_\alpha(aa_0) - \rho_\alpha(a_0a) + \rho_\alpha(a_0a) - \phi(a)\rho_\alpha(a_0) \\ &\rightarrow 0 \end{aligned}$$

for every $a \in A$. Also we have

$$\phi \circ \pi_A(n_\alpha) - 1 = \phi \circ \pi_A \circ \rho_\alpha(a_0) - \phi(a_0) \rightarrow 0.$$

Define $T : A \otimes_p A \rightarrow A$ by $T(a \otimes b) = \phi(b)T(a)$ for each $a \in A$ and $b \in B$. It is clear that T is a bounded linear map which satisfies

$$T(a \cdot x) = aT(x), \quad T(x \cdot a) = \phi(T(x)), \quad \phi \circ T = \phi \circ \pi_A, \quad (a \in A, x \in A \otimes_p A).$$

Set $m_\alpha = T(n_\alpha)$. One can show that

$$aT(n_\alpha) - \phi(a)T(n_\alpha) = T(a \cdot n_\alpha - \phi(a)n_\alpha) \rightarrow 0, \quad (a \in A)$$

and

$$\phi(m_\alpha) = \phi \circ T(n_\alpha) = \phi \circ \pi_A(n_\alpha) \rightarrow 1.$$

Thus A is approximately left ϕ -amenable. \square

Proposition 2.2. *Let A be a Banach algebra with $\phi \in \Delta(A)$. If A is approximately biprojective, then A is approximately left ϕ -biprojective.*

Proof. Since A is approximately biprojective, there exists a net of A -bimodule morphism $\rho_\alpha : A \rightarrow A \otimes_p A$ such that

$$\pi_A \circ \rho_\alpha(a) \rightarrow a \quad (a \in A).$$

Pick $a_0 \in A$ such that $\phi(a_0) = 1$. Set $T : A \otimes_p A \rightarrow A \otimes_p A$ which defined by $T(a \otimes b) = \phi(a)a_0 \otimes b$ for each $a, b \in A$. It is easy to see that

$$\phi(x)T(a \otimes b) = \phi(x)\phi(a)a_0 \otimes b = \phi(xa)a_0 \otimes b = T(x \cdot a \otimes b)$$

and

$$\phi \circ \pi_A \circ T(a \otimes b) = \phi(\phi(a)a_0b) = \phi(ab) = \phi \circ \pi_A(a \otimes b)$$

for each $a, b, x \in A$. Using these facts one can see that $(T \circ \rho_\alpha)_\alpha$ satisfies the conditions in Definition 1.1. So A is approximately left ϕ -biprojective. \square

There exists a Banach algebra which is never approximately left ϕ -biprojective.

Example 2.3. Consider a Triangular Banach algebra $T = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. Define $\phi \in \Delta(T)$ by $\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = c$ for all $a, b, c \in \mathbb{C}$. We claim that T is not approximately ϕ -biprojective. To see this we go toward a contradiction and assume that T is approximately left ϕ -biprojective. Since T is unital, by Proposition 2.1 T is approximately left ϕ -amenable. Set $I = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. It is easy to see that $\phi|_I \neq 0$ then by [18, Proposition 5.1] I is approximately left ϕ -amenable. Thus there exists a net (i_α) in I such that

$$ii_\alpha - \phi(i)i_\alpha \rightarrow 0, \quad \phi(i_\alpha) \rightarrow 1, \quad (i \in I).$$

Hence there exist net (a_α) and (b_α) in \mathbb{C} such that $i_\alpha = \begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix}$. So for each $i = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ in I , we have

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix} - b \begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix} \rightarrow 0.$$

Then $ab_\alpha - ba_\alpha \rightarrow 0$, for each $a, b \in \mathbb{C}$. Since $b_\alpha \rightarrow 1$, taking $a = 1$ and $b = 0$, gives a contradiction.

We remind that by [1, Proposition 2.7], A is approximately left ϕ -amenable if and only if there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that

$$a \cdot m_\alpha - \phi(a)m_\alpha \rightarrow 0, \quad \tilde{\phi} \circ \pi_A^{**}(m_\alpha) \rightarrow 1,$$

for all $a \in A$.

Proposition 2.4. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is approximately left ϕ -amenable, then A is approximately left ϕ -biprojective.*

Proof. Let A be approximately left ϕ -amenable. Then there exists a net m_α in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha - \phi(a)m_\alpha \rightarrow 0$ and $\tilde{\phi} \circ \pi_A^{**}(m_\alpha) = 1$, for each $a \in A$, see [1, Proposition 2.7]. Take $\epsilon > 0$, $F \subseteq A$ and $\Lambda \subseteq (A \otimes_p A)^*$ arbitrary finite subsets. Then we have

$$\|a \cdot m_\alpha - \phi(a)m_\alpha\| < \epsilon, \quad |\tilde{\phi} \circ \pi_A^{**}(m_\alpha) - 1| < \epsilon, \quad (a \in F).$$

It is well-known that for each α , there exists a net $(n_\beta^\alpha)_\beta$ in $A \otimes_p A$ such that $n_\beta^\alpha \xrightarrow{w^*} m_\alpha$. Since π_A^{**} is a w^* -continuous map, then

$$\pi_A(n_\beta^\alpha) = \pi_A^{**}(n_\beta^\alpha) \xrightarrow{w^*} \pi_A^{**}(m_\alpha).$$

Thus we have

$$|a \cdot n_\beta^\alpha(f) - a m_\alpha(f)| < \frac{\epsilon}{K_0}, \quad |\phi(a)n_\beta^\alpha(f) - \phi(a)m_\alpha(f)| < \frac{\epsilon}{K_0}, \quad |\phi \circ \pi_A(n_\beta^\alpha) - \tilde{\phi} \circ \pi_A^{**}(m_\alpha)| < \epsilon, \quad (a \in F, f \in \Lambda),$$

where $K_0 = \sup\{\|f\| \mid f \in \Lambda\}$. Since $a \cdot m_\alpha - \phi(a)m_\alpha \rightarrow 0$ and $\tilde{\phi} \circ \pi_A^{**}(m_\alpha) = 1$, we can find $\beta = \beta(F, \Lambda, \epsilon)$ such that

$$|a \cdot n_\beta^\alpha(f) - \phi(a)n_\beta^\alpha(f)| < c \frac{\epsilon}{K_0}, \quad |\phi \circ \pi_A(n_\beta^\alpha) - 1| < \epsilon, \quad (a \in F, f \in \Lambda)$$

for some $c \in \mathbb{R}$. Using Mazur's lemma, we have a net $(n_{(F, \Lambda, \epsilon)})$ in $A \otimes_p A$ such that

$$\|a \cdot n_{(F, \Lambda, \epsilon)} - \phi(a)n_{(F, \Lambda, \epsilon)}\| \rightarrow 0, \quad |\phi \circ \pi_A(n_{(F, \Lambda, \epsilon)}) - 1| \rightarrow 0, \quad (a \in F).$$

Define $\rho_{(F, \Lambda, \epsilon)} : A \rightarrow A \otimes_p A$ by $\rho_{(F, \Lambda, \epsilon)}(a) = a \cdot n_{(F, \Lambda, \epsilon)}$ for each $a \in A$. It is clear that $\rho_{(F, \Lambda, \epsilon)}(ab) = a \cdot \rho_{(F, \Lambda, \epsilon)}(b)$ for each $a, b \in A$.

$$(2.1) \quad \begin{aligned} \|\rho_{(F, \Lambda, \epsilon)}(ab) - \phi(b)\rho_{(F, \Lambda, \epsilon)}(a)\| &= \|ab \cdot n_{(F, \Lambda, \epsilon)} - \phi(b)a \cdot n_{(F, \Lambda, \epsilon)}\| \\ &\leq \|a\| \|b \cdot n_{(F, \Lambda, \epsilon)} - \phi(b)n_{(F, \Lambda, \epsilon)}\| \rightarrow 0, \end{aligned}$$

for each $a, b \in A$. Also

$$(2.2) \quad \begin{aligned} |\phi \circ \pi_A \circ \rho_{(F, \Lambda, \epsilon)}(a) - \phi(a)| &= |\phi \circ \pi_A(a \cdot n_{(F, \Lambda, \epsilon)}) - \phi(a)| = |\phi(a)| |\phi \circ \pi_A(n_{(F, \Lambda, \epsilon)}) - 1| \\ &\rightarrow 0, \end{aligned}$$

for each $a \in A - \ker \phi$. It is easy to see that $\phi \circ \pi_A \circ \rho_{(F, \Lambda, \epsilon)}(a) = \phi(a)$ for each $a \in \ker \phi$. \square

Remark 2.5. Let A be a Banach algebra and $\phi \in \Delta(A)$. Using the argument of previous Theorem one can see that if A is either pseudo-amenable or approximately amenable, then A is approximately left ϕ -biprojective.

Theorem 2.6. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is ϕ -biflat, then A is approximately left ϕ -biprojective.*

Proof. Since A is ϕ -biflat then there exists a bounded A -bimodule morphism $\rho : A \rightarrow (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for each $a \in A$. There exists a net (ρ_α) in $B(A, A \otimes_p A)$ (the set of bounded linear maps from A into $A \otimes_p A$) such that ρ_α converges to ρ in the *weak-star operator topology*. Since π_A^{**} is a w^* -continuous map, for each $a \in A$ we have

$$\pi_A \circ \rho_\alpha(a) = \pi_A^{**} \circ \rho_\alpha(a) \xrightarrow{w^*} \pi_A^{**} \circ \rho(a),$$

so

$$\phi \circ \pi_A \circ \rho_\alpha(a) \rightarrow \tilde{\phi} \circ \pi_A^{**} \circ \rho(a).$$

Let $\epsilon > 0$ and take $F = \{a_1, a_2, \dots, a_r\}$ and $G = \{x_1, x_2, \dots, x_r\}$ arbitrary finite subsets of A . Define

(2.3)

$$M = \{(a_1 \cdot T(x_1) - T(a_1 x_1), a_2 \cdot T(x_2) - T(a_2 x_2), \dots, a_r \cdot T(x_r) - T(a_r x_r)),$$

$$\phi \circ \pi_A \circ T(x_i) - \phi(x_i) | T \in B(A, A \otimes_p A)\}_{i=1, \dots, r} \subseteq \prod_{i=1}^r (A \otimes_p A) \oplus_1 \mathbb{C}.$$

It is clear that M is a convex set and $(0, 0, \dots, 0)$ belongs to \overline{M}^w . Using Mazur's Lemma $(0, 0, \dots, 0) \in \overline{M}^w = \overline{M}^{||\cdot||}$. Then we can find an element $\theta_{(F, G, \epsilon)}$ in $B(A, A \otimes_p A)$ such that

$$\|a_i \cdot \theta_{(F, G, \epsilon)}(b_i) - \theta_{(F, G, \epsilon)}(a_i b_i)\| < \epsilon, \quad \|\theta_{(F, G, \epsilon)}(a_i b_i) - \theta_{(F, G, \epsilon)}(a_i) \cdot b_i\| < \epsilon$$

and

$$|\phi \circ \pi_A \circ \theta_{(F, G, \epsilon)}(a_i) - \phi(a_i)| < \epsilon,$$

for each $i \in \{1, 2, \dots, r\}$. Hence the net $(\theta_{(F, G, \epsilon)})_{(F, G, \epsilon)}$ satisfies

$$a \cdot \theta_{(F, G, \epsilon)}(b) - \theta_{(F, G, \epsilon)}(ab) \rightarrow 0, \quad \theta_{(F, G, \epsilon)}(ab) - \theta_{(F, G, \epsilon)}(a) \cdot b \rightarrow 0$$

and $\phi \circ \pi_A \circ \theta_{(F, G, \epsilon)}(a) - \phi(a) \rightarrow 0$ for each $a, b \in A$. Set T the same map as in the proof of Proposition 2.2. It is easy to see that $(T \circ \theta_{(F, G, \epsilon)})_{(F, G, \epsilon)}$ satisfies the conditions in Definition 1.1. So A is approximately left ϕ -biprojective. \square

We have to remind that every biflat Banach algebra A with $\phi \in \Delta(A)$ is ϕ -biflat. Then Using previous Theorem, we have the following corollary:

Corollary 2.7. *Suppose that A is a biflat Banach algebra with $\phi \in \Delta(A)$. Then A is approximately left ϕ -biprojective.*

Proposition 2.8. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that I is closed ideal of A which $\phi|_I \neq 0$. If A is approximately left ϕ -biprojective, then I is approximately left ϕ -biprojective.*

Proof. Let $(\rho_\alpha)_\alpha$ be a net of maps which satisfies Definition 1.1. Take i_0 in I such that $\phi(i_0) = 1$. Define $T : A \otimes_p A \rightarrow I \otimes_p I$ by $T(a \otimes b) = a i_0 \otimes i_0 b$ for each $a, b \in A$. It is easy to see that T is a bounded linear map. Set $\eta_\alpha = T \circ \rho_\alpha|_I : I \rightarrow I \otimes_p I$. Then we have

$$i \cdot \eta_\alpha(j) - \eta_\alpha(ij) = T(i \cdot \rho_\alpha(j) - \rho_\alpha(ij)) \rightarrow 0$$

and

$$\eta_\alpha(ij) - \phi(j)\eta_\alpha(i) = T(\rho_\alpha(ij) - \phi(j)\rho_\alpha(i)) \rightarrow 0$$

also

$$\phi \circ \pi_I \circ \eta_\alpha(i) - \phi(i) = \phi \circ \pi_I \circ T \circ \rho_\alpha(i) - \phi(i) = \phi \circ \pi_A \circ \rho_\alpha(i) - \phi(i) \rightarrow 0,$$

for each $i, j \in I$. □

Theorem 2.9. *Let A and B be Banach algebras and $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Suppose that A is unital and B has an idempotent x_0 such that $\psi(x_0) = 1$. If $A \otimes_p B$ is approximately left $\phi \otimes \psi$ -biprojective, then A is approximately left ϕ -biprojective.*

Proof. Let $(\rho_\alpha) : A \otimes_p B \rightarrow (A \otimes_p B) \otimes_p (A \otimes_p B)$ be a net of continuous maps such that

$$x \cdot \rho_\alpha(y) - \rho_\alpha(xy) \rightarrow 0, \quad \rho_\alpha(xy) - \phi(x)\rho_\alpha(y) \rightarrow 0$$

and

$$\phi \otimes \psi \circ \pi_{A \otimes_p B}(x) - \phi \otimes \psi(x) \rightarrow 0,$$

for each $x, y \in A \otimes_p B$. Note that $A \otimes_p B$ with the following actions becomes a Banach A -bimodule:

$$a_1 \cdot (a_2 \otimes b) = a_1 a_2 \otimes b, \quad (a_2 \otimes b) \cdot a_1 = a_2 a_1 \otimes b, \quad (a_1, a_2 \in A, b \in B).$$

Consider

$$\begin{aligned} \rho_\alpha(a_1 a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) + \\ &\quad (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) + \\ &\quad (a_1 \cdot (e \otimes x_0)) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) + \\ &\quad (a_1 \cdot (e \otimes x_0)) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(e a_2 \otimes x_0 x_0) + \\ &\quad a_1 \cdot \rho_\alpha(e a_2 \otimes x_0 x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \rho_\alpha(a_1 a_2 \otimes x_0) - \phi(a_2)\rho_\alpha(a_1 \otimes x_0) &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - \phi(a_2)\rho_\alpha(a_1 \otimes x_0) \\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - \phi \otimes \psi(a_2 \otimes x_0)\rho_\alpha(a_1 \otimes x_0) + \\ &\quad \phi \otimes \psi(a_2 \otimes x_0)\rho_\alpha(a_1 \otimes x_0) - \phi(a_2)\rho_\alpha(a_1 \otimes x_0) \rightarrow 0, \end{aligned}$$

for each $a_1, a_2 \in A$. Define $T : (A \otimes_p B) \otimes_p (A \otimes_p B) \rightarrow A \otimes_p A$ by $T((a \otimes b) \otimes (c \otimes d)) = \psi(bd)a \otimes c$, for each $a, c \in A, b, d \in B$. One can see that T is a bounded linear operator and $\pi_A \circ T = (id \otimes \psi) \circ \pi_{A \otimes_p B}$, where $id \otimes \psi(a \otimes b) = \psi(b)a$ for all $a \in A, b \in B$. Set $\eta_\alpha(a) = T \circ \rho_\alpha(a \otimes x_0)$. It is easy to see that for each $a, \eta_\alpha : A \rightarrow A \otimes_p A$ is a continuous map which satisfies

$$a \cdot \eta_\alpha(b) - \eta_\alpha(ab) \rightarrow 0, \quad \eta_\alpha(ab) - \phi(b)\eta_\alpha(a) \rightarrow 0, \quad (a, b \in A).$$

Also we have

$$\phi \circ \pi_A \circ \eta_\alpha(a) = \phi \circ \pi_A \circ T \circ \rho_\alpha(a \otimes x_0) = \phi \circ (id \otimes \psi) \circ \pi_{A \otimes_p B} \circ \rho_\alpha(a \otimes x_0) \rightarrow \phi(a),$$

for each $a \in A$. Hence A is approximately left ϕ -biprojective. □

3. APPLICATION TO BANACH ALGEBRAS ASSOCIATED WITH A LOCALLY COMPACT GROUP

Let G be a locally compact group and let \widehat{G} be its dual group, which consists of all non-zero continuous homomorphism $\zeta : G \rightarrow \mathbb{T}$. It is well-known that $\Delta(L^1(G)) = \{\phi_\zeta : \zeta \in \widehat{G}\}$, where $\phi_\zeta(f) = \int_G \overline{\zeta(x)} f(x) dx$ and dx is a left Haar measure on G , for more details, see [9, Theorem 23.7].

The map $\phi_1 : L^1(G) \rightarrow \mathbb{C}$ which is specified by

$$\phi_1(f) = \int_G f(x) dx$$

is called augmentation character. We know that augmentation character induce a character on $S(G)$ which we denote it by ϕ_1 again, see [2].

We recall that, for a locally compact group G , a linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G if it satisfies the following properties:

- (i) $S(G)$ is a dense left ideal in $L^1(G)$;
- (ii) $S(G)$ with respect to some norm $\|\cdot\|_{S(G)}$ is a Banach space and $\|f\|_{L^1(G)} \leq \|f\|_{S(G)}$;
- (iii) For $f \in S(G)$ and $y \in G$, $L_y f \in S(G)$ and the map $y \mapsto \delta_y * f$ is continuous. Also $\|\delta_y * f\|_{S(G)} = \|f\|_{S(G)}$, for $f \in S(G)$ and $y \in G$.

For more information about this algebras see [15].

Theorem 3.1. *Let G be a locally compact SIN-group. If $S(G)$ is approximately ϕ_1 -biprojective, then G is amenable.*

Proof. Using the main result of [13], G is a SIN group if and only if $S(G)$ has a central approximate identity. Then we have an element $a_0 \in S(G)$ such that $aa_0 = a_0a$ and $\phi_1(a_0) = 1$ for each $a \in S(G)$. Applying Proposition 2.1, approximate left ϕ_1 -biprojectivity of $S(G)$ implies that $S(G)$ is approximately left ϕ_1 -amenable. We can find a net (m_α) in $S(G)$ such that

$$\|am_\alpha - \phi_1(a)m_\alpha\|_{S(G)} \rightarrow 0, \quad \phi_1(m_\alpha) \rightarrow 1 \quad (a \in S(G)).$$

Since $\|\cdot\|_{L^1(G)} \leq \|\cdot\|_{S(G)}$, then

$$\|am_\alpha - \phi_1(a)m_\alpha\|_{L^1(G)} \rightarrow 0, \quad \phi_1(m_\alpha) \rightarrow 1 \quad (a \in S(G)).$$

Let f be an element of $S(G)$ such that $\phi_1(f) = 1$. Define $f_\alpha = fm_\alpha$. For each $y \in G$ we have

$$\phi_1(\delta_y f) = \int_G \delta_y f(x) dx = \int_G f(y^{-1}x) dx = \int_G f(x) dx = \phi_1(f),$$

where δ_y denotes the point mass at $\{y\}$. We have

$$\begin{aligned} \|\delta_y f_\alpha - f_\alpha\|_{L^1(G)} &= \|(\delta_y f)m_\alpha - fm_\alpha\|_{L^1(G)} \\ &\leq \|(\delta_y f)m_\alpha - m_\alpha\|_{L^1(G)} + \|m_\alpha - fm_\alpha\|_{L^1(G)} \\ (3.1) \quad &\leq \|(\delta_y f)m_\alpha - \phi_1(\delta_y f)m_\alpha\|_{L^1(G)} + \|\phi_1(\delta_y f)m_\alpha - m_\alpha\|_{L^1(G)} \\ &\quad + \|m_\alpha - \phi_1(f)m_\alpha\|_{L^1(G)} + \|\phi_1(f)m_\alpha - fm_\alpha\|_{L^1(G)} \\ &\rightarrow 0. \end{aligned}$$

On the other hand

$$\phi_1(f_\alpha) = \phi_1(fm_\alpha) = \phi_1(f)\phi_1(m_\alpha) \rightarrow 1.$$

Since $|\phi_1(f_\alpha)| \leq \|f_\alpha\|_{L^1(G)}$, then (f_α) can stay away from 0. Without loss of generality we assume that $\|f_\alpha\|_{L^1(G)} \geq \frac{1}{2}$. Define $g_\alpha = \frac{|f_\alpha|}{\|f_\alpha\|_{L^1(G)}}$. It is clear that (g_α) is a bounded net in $L^1(G)$. Consider

$$\|\delta_y g_\alpha - g_\alpha\|_{L^1(G)} \leq 2\|\delta_y |f_\alpha| - |f_\alpha|\|_{L^1(G)} \leq 2\|\delta_y f_\alpha - f_\alpha\|_{L^1(G)} \rightarrow 0.$$

Now by [17, Exercise 1.1.6], G is amenable. \square

We give a Banach algebra related to a locally compact group which is never approximately left ϕ -biprojective.

Example 3.2. Let G be a locally compact group and let $A(G)$ be the Fourier algebra with respect to G . Let $T = \begin{pmatrix} A(G) & A(G) \\ 0 & A(G) \end{pmatrix}$ be a Triangular Banach algebra related to $A(G)$. Suppose that $\phi \in \Delta(A(G))$.

Define $\psi_\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \phi(c)$ for all $a, b, c \in A(G)$. It is easy to see that $\psi_\phi \in \Delta(T)$. Note that $A(G)$ is a commutative Banach algebra, hence there exists $a_0 \in A(G)$ such that $aa_0 = a_0a$ and $\phi(a_0) = 1$ for each $a \in A(G)$. Set $t_0 = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}$, clearly $tt_0 = t_0t$, $\psi_\phi(t_0) = 1$, for every $t \in T$. Using Proposition 2.1, we have T is approximately left ψ_ϕ -amenable. Proceed a similar arguments as in the Example 2.3, we have a net (a_α) in $A(G)$ such that $a - ba_\alpha \rightarrow 0$ for each $a, b \in A(G)$. By taking $a \in A(G)$ such that $\phi(a) = 1$ and $b \in \ker \phi$, we have $\phi(a) = \phi(a) - \phi(b)\phi(a_\alpha) = \phi(a - ba_\alpha) \rightarrow 0$ which is a contradiction.

Lemma 3.3. *Let G be a locally compact group. Then $A(G)$ is approximately left ϕ -biprojective.*

Proof. By [12, Example 2.6] $A(G)$ is left ϕ -amenable for each $\phi \in \Delta(G)$. Then is approximately left ϕ -amenable. Applying Proposition 2.4 implies that $A(G)$ is approximately left ϕ -biprojective, for each $\phi \in \Delta(A(G))$. \square

Let G be a locally compact group and let $M(G)$ be the measure algebra with respect to G . It is well-known that $L^1(G)$ is a closed ideal of $M(G)$. So every character of $L^1(G)$ has an extension to $M(G)$, particularly the augmentation character ϕ_1 . We again denote this extension by ϕ_1 .

Theorem 3.4. *Let G be a locally compact group. $M(G)$ is approximately left ϕ_1 -biprojective if and only if G is amenable.*

Proof. Suppose that $M(G)$ is approximately left ϕ_1 -biprojective. Since $M(G)$ is unital, by Proposition 2.1 $M(G)$ is approximately left ϕ_1 -amenable. Note that $L^1(G)$ is a closed ideal of $M(G)$ and $\phi_1|_{L^1(G)} \neq 0$ so by [12, Lemma 3.1] $L^1(G)$ is approximately left ϕ_1 -amenable. Using similar method as in the proof of Theorem 3.1 G is amenable.

For converse, let G be an amenable group. By Johnson theorem $L^1(G)$ is amenable. Hence $L^1(G)$ is left ϕ_1 -amenable. Hence $M(G)$ is left ϕ_1 -amenable. So $M(G)$ is approximately left ϕ_1 -amenable. Using Proposition 2.4 implies that $M(G)$ is approximately left ϕ_1 -biprojective. \square

Corollary 3.5. *Let G be a locally compact group. $M(G)$ is approximately left character biprojective if and only if G is discrete and amenable.*

Proof. Let G be a locally compact group. Suppose that $M(G)$ is approximately left character biprojective. Since $M(G)$ is unital, then by Proposition 2.1 approximately character biprojectivity implies that $M(G)$ is approximately character amenable. Applying [1, Theorem 7.2] G is discrete and amenable.

For converse, let G be amenable and discrete. Then by [7, Proposition 4.2] $M(G)$ is pseudo-amenable. Hence by Remark 2.5, $M(G)$ is approximately character left biprojective. \square

We give a Banach algebra which is not pseudo-amenable but is approximately left ϕ -biprojective.

Example 3.6. Let G be an infinite compact group. It is well-known that for $p \geq 1$, $\Delta(L^p(G)) = \{\phi_\rho | \rho \in \widehat{G}\}$, where \widehat{G} is the dual group of G and $\phi_\rho(f) = \int_G f(x) \overline{\rho(x)} dx$, see [9]. Since G is compact $\widehat{G} \subseteq L^\infty(G) \subseteq L^p(G)$. It is easy to see that

$$f\rho = \phi_\rho(f)\rho, \quad \phi_\rho(\rho) = \int_G \rho(x) \overline{\rho(x)} dx = \int_G 1 dx = 1, \quad (f \in L^1(G))$$

(we assume that dx is the normalized left Haar measure on G). Since $\rho \in L^p(G)$, then the map $f \mapsto f\rho$ is w^* -continuous on $L^p(G)^{**}$. Hence we have

$$f\rho = \tilde{\phi}_\rho(f)\rho, \quad \phi_\rho(\rho) = \tilde{\phi}_\rho(\rho) = 1, \quad (f \in L^p(G)^{**}).$$

It means that $L^p(G)^{**}$ is left $\tilde{\phi}_\rho$ -amenable, so $L^p(G)^{**}$ is approximately left $\tilde{\phi}_\rho$ -amenable. Therefore by Proposition 2.4, $L^p(G)^{**}$ is approximately left $\tilde{\phi}_\rho$ -biprojective. Particularly $L^1(G)^{**}$ is approximately left $\tilde{\phi}_\rho$ -biprojective but if $L^1(G)^{**}$ is pseudo-amenable, then by [7, Proposition 4.2] G is discrete and amenable. Since G is compact, then G must be finite which is a contradiction.

Theorem 3.7. *Let G be a locally compact SIN group. $L^1(G)^{**}$ is approximately left character biprojective if and only if G is amenable.*

Proof. Suppose that $L^1(G)^{**}$ is approximately left character biprojective. Since G is a SIN group, then by the main result of [13], $L^1(G)$ has a central approximate identity. Then for each $\phi \in \Delta(L^1(G))$ there exists an element $a_0 \in L^1(G)$ such that $aa_0 = a_0a$ and $\phi(a_0) = 1$, for each $a \in L^1(G)$. Since for each $a \in L^1(G)$ two maps $b \mapsto ab$ and $a \mapsto ba$ is w^* -continuous on $L^1(G)^{**}$, we have

$$aa_0 = a_0a, \quad \phi(a_0) = \tilde{\phi}(a_0) = 1 \quad (a \in L^1(G)^{**}).$$

Using Proposition 2.4, implies that $L^1(G)^{**}$ is approximately left ϕ -amenable for all $\phi \in \Delta(L^1(G)^{**})$. By [1, Proposition 3.9] $L^1(G)$ is approximately left ϕ -amenable. Hence [1, Theorem 7.1] implies that G is amenable.

For converse, suppose that G is amenable. So by Johnson theorem, $L^1(G)$ is amenable, hence $L^1(G)$ is left ϕ -amenable. By [12, Proposition 3.4] we have $L^1(G)^{**}$ is left $\tilde{\phi}$ -amenable for all $\phi \in \Delta(L^1(G))$. Hence $L^1(G)^{**}$ is approximately left $\tilde{\phi}$ -amenable for all $\phi \in \Delta(L^1(G))$. Now by Theorem 2.4 $L^1(G)^{**}$ is approximately left character biprojective. \square

It is well-known that for each semigroup S there exists a partial order on $E(S)$, where $E(S)$ is the set of idempotents of S . Indeed

$$s \leq t \Leftrightarrow s = st = ts \quad (s, t \in E(S)).$$

The semigroup S is called inverse semigroup, if for each $s \in S$ there exists $s^* \in S$ such that $ss^*s = s^*$ and $s^*ss^* = s$ for each $s \in S$. Inverse semigroup S is called Clifford semigroup if for each $s \in S$ there exists $s^* \in S$ such that $ss^* = s^*s$. There exists a partial order on each inverse semigroup S , that is,

$$s \leq t \Leftrightarrow s = ss^*t \quad (s, t \in S).$$

Note that these two partial orders on an inverse semigroup are the same. Let (S, \leq) be an inverse semigroup. For each $s \in S$, set $[x] = \{y \in S \mid y \leq x\}$. S is called uniformly locally finite if $\sup\{|[x]| < \infty \mid x \in S\}$. Suppose that S is an inverse semigroup and $e \in E(S)$. $G_e = \{s \in S \mid ss^* = s^*s = e\}$ is a maximal subgroup of S with respect to e . For more information about semigroup theory see [10].

Theorem 3.8. *Let $S = \cup_{e \in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. $\ell^1(S)$ is approximately left character biprojective if and only if $\ell^1(S)$ pseudo-amenable.*

Proof. Suppose that $\ell^1(S)$ is approximately left character biprojective. By [16, Theorem 2.16], $\ell^1(S) \cong \ell^1 - \oplus_{e \in E(S)} \ell^1(G_e)$. Since $\ell^1(G_e)$ has a character ϕ_1 (at least augmentation character), then this character extends on $\ell^1(S)$ which we denote this extension with ϕ_1 again. So $\ell^1(S)$ is approximately left ϕ_1 -biprojective. Since $\phi_1|_{\ell^1(G_e)} \neq 0$ and $\ell^1(G_e)$ is a closed ideal of $\ell^1(S)$, by Proposition 2.8, $\ell^1(G_e)$ is approximately left ϕ_1 -biprojective. On the other hand since $\ell^1(G_e)$ is unital, by Proposition 2.1, $\ell^1(G_e)$ is approximately left ϕ_1 -amenable. So by [1, Theorem 7.1], G_e is amenable for all $e \in E(S)$. Thus by [4, Corollary 3.9] $\ell^1(S)$ is pseudo-amenable.

Converse is true by Remark 2.5. □

4. EXAMPLES

We give a Banach algebra which is approximately left ϕ -biprojective but is not ϕ -biprojective.

Remark 4.1. Consider the semigroup \mathbb{N}_\vee , with semigroup operation $m \vee n = \max\{m, n\}$, where m and n are in \mathbb{N} . The character space $\Delta(\ell^1(\mathbb{N}_\vee))$ precisely consists of the all functions $\phi_n : \ell^1(\mathbb{N}_\vee) \rightarrow \mathbb{C}$ defined by $\phi_n(\sum_{i=1}^\infty \alpha_i \delta_i) = \sum_{i=1}^n \alpha_i$ for every $n \in \mathbb{N} \cup \{\infty\}$. For more information about this semigroup algebra see [3]. In [21], author with A. Pourabbas showed that $\ell^1(\mathbb{N}_\vee)$ is ϕ_n -biflat for each $n \in \mathbb{N} \cup \{\infty\}$. Since this algebra is commutative, by [21, Proposition 3.3] $\ell^1(\mathbb{N}_\vee)$ is left ϕ_n -amenable. Then $\ell^1(\mathbb{N}_\vee)$ is approximately left ϕ_n -amenable. By Proposition 2.4, $\ell^1(\mathbb{N}_\vee)$ is approximately character left biprojective. Hence $\ell^1(\mathbb{N}_\vee)$ is approximately left ϕ_∞ biprojective. Moreover we showed that $\ell^1(\mathbb{N}_\vee)$ is ϕ_n -biprojective for each $n \in \mathbb{N}$. But if $\ell^1(\mathbb{N}_\vee)$ is ϕ_∞ -biprojective, then $\ell^1(\mathbb{N}_\vee)$ is character biprojective. So by [19, Remark 3.6] and [19, Lemma 3.7], the maximal ideal space of $\ell^1(\mathbb{N}_\vee)$ is finite, which is impossible because the maximal ideal space of $\ell^1(\mathbb{N}_\vee)$ is $\mathbb{N} \cup \{\infty\}$.

We give a Banach algebra which is neither left ϕ -amenable nor ϕ -biflat but is approximately left ϕ -biprojective. Hence the converse of Theorem 2.6 is not always true.

Example 4.2. We denote ℓ^1 for the set of all sequences $a = ((a_n))$ of complex numbers with $\|a\| = \sum_{n=1}^\infty |a_n| < \infty$. Equip ℓ^1 with the following product:

$$(a * b)(n) = \begin{cases} a(n)b(n) & n = 1 \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & n > 1, \end{cases}$$

and $\|\cdot\|$ becomes a Banach algebra. It is easy to see that $\Delta(\ell^1) = \{\phi_1, \phi_1 + \phi_n\}$, where $\phi_n(a) = a(n)$ for each $a \in \ell^1$. By [14, Example 2.9] ℓ^1 is not left ϕ_1 -amenable. Suppose conversely that A is ϕ -biflat. Since ℓ^1 is commutative, by [21, Proposition 3.3] ϕ -biflatness follows that ℓ^1 is left ϕ_1 -amenable, which is a contradiction.

Using [14, Example 2.9], ℓ^1 is approximately left ϕ_1 -amenable. Then Proposition 2.4 implies that ℓ^1 is approximately left ϕ_1 -biprojective. Moreover [14, Example 2.9] showed that ℓ^1 is left $\phi_1 + \phi_n$ -amenable so ℓ^1 is approximately left $\phi_1 + \phi_n$ -biprojective. Hence ℓ^1 is approximately left character biprojective.

We give a Banach algebra which is approximately left ϕ -biprojective but is not approximately left ϕ -amenable. Then the converse of Proposition 2.4 is not always true.

Example 4.3. Let S be a left zero semigroup with $|S| \geq 2$, that is a semigroup with product $st = s$ for all $s, t \in S$. For the semigroup algebra $\ell^1(S)$, we have $fg = \phi_S(g)f$, where ϕ_S is the augmentation character on $\ell^1(S)$. We claim that $\ell^1(S)$ is approximately left ϕ_S -biprojective. To see this, let $f_0 \in \ell^1(S)$ be an element such that $\phi_S(f_0) = 1$. Define $\rho : \ell^1(S) \rightarrow \ell^1(S) \otimes_p \ell^1(S)$ by $\rho(f) = f \otimes f_0$ for all $f \in \ell^1(S)$. It is easy to see that

$$f \cdot \rho(g) = \rho(fg), \quad \rho(fg) = \phi_S(g)\rho(f), \quad \phi_S \circ \pi_A \circ \rho(f) = \phi_S(f_0 f) = \phi(f), \quad (f, g \in \ell^1(S)).$$

We show that $\ell^1(S)$ is not approximately left ϕ -amenable, provided that $|S| \geq 2$. We go toward a contradiction and suppose that $\ell^1(S)$ is approximately left ϕ -amenable. Then there exists a net (f_α) in $\ell^1(S)$ such that

$$\phi_S(f_\alpha) = 1, \quad ff_\alpha - \phi_S(f)f_\alpha \rightarrow 0 \quad (f \in \ell^1(S)).$$

It follows that $f - \phi_S(f)f_\alpha \rightarrow 0$ for each $f \in \ell^1(S)$. Since S has at least two elements s_1, s_2 , take $f = \delta_{s_1}$ and $f = \delta_{s_2}$ and put it in $f - \phi_S(f)f_\alpha \rightarrow 0$. It follows that $\delta_{s_1} = \delta_{s_2}$, so $s_1 = s_2$ which is impossible.

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